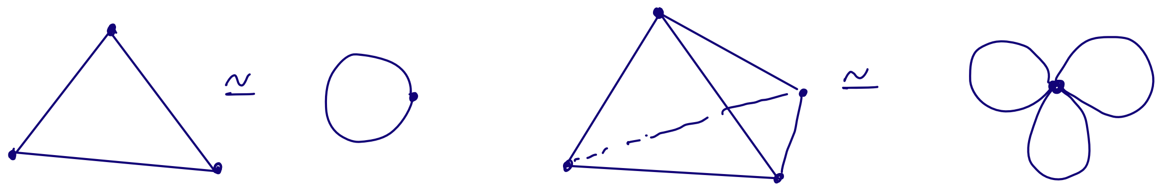
(I) Overview

Idea: To study the homotopy type of a simplicial complex via "nice" functions on it.

Eg:



Defn (Discrete Morse Functions)

Let K = finite simplicial complex.

f: $\mathbb{R}$-valued function on the simplices of X

s.t., for every simplex σ,

$\tau \supset \sigma \implies f(\tau) > f(\sigma)$

$\sigma \supset \tau \implies f(\sigma) > f(\tau)$

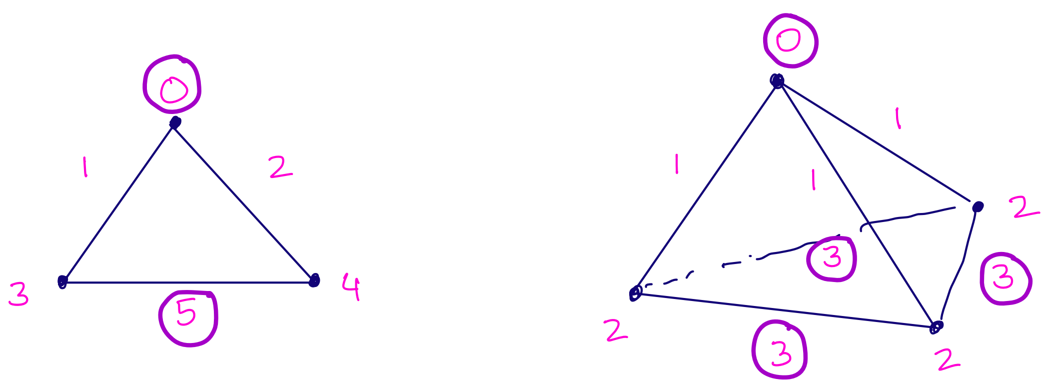
One exception allowed

Defn (Critical Simplices)

σ is critical if $f(\tau) > f(\sigma)\ \forall\ \tau \supset \sigma$ and

$f(\tau) < f(\sigma)\ \forall\ \tau \subset \sigma$

Eg:



Thm: If f is a Discrete Morse Function, then K is htpy equivalent to a CW-complex which has a p-cell for every critical p-simplex of K.

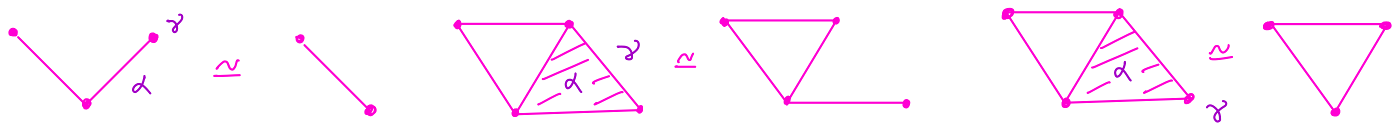
(II) Tools for Proof

(1) Simplicial Collapse

K : simplicial complex. $\gamma \subset \alpha$ simplices of K s.t.
γ is a "free face" (i.e. $\gamma \subsetneq \tau \Rightarrow \tau = \alpha$)

Let $L = K \setminus \{\gamma, \alpha\}$.

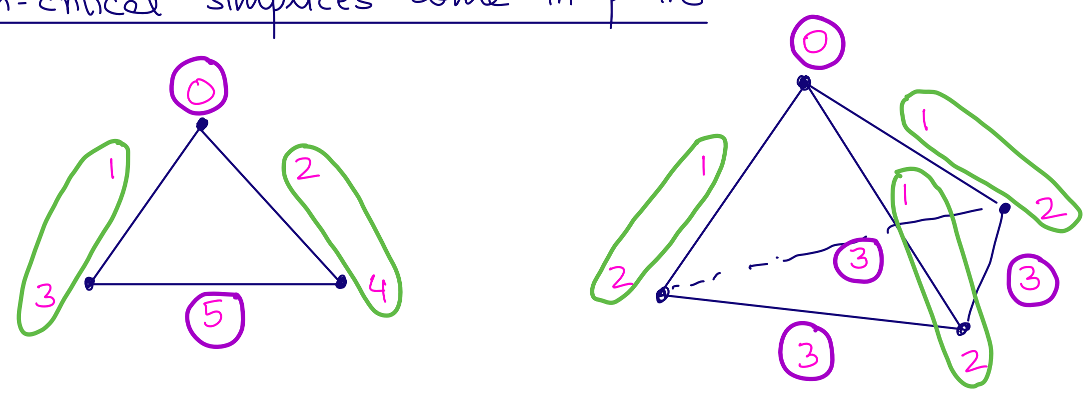
We say K simplicially collapses to L



The proof of the theorem is via a suitably chosen sequence of simplicial collapses.

(2) Non-critical simplices come in pairs

Eg:

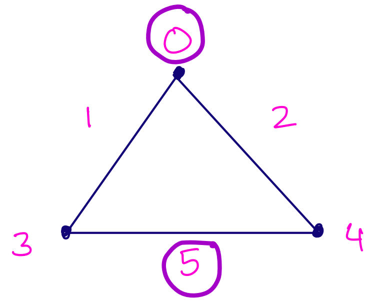


(3) Proof Sketch

1. Assume all f-values are distinct.
2. For $c \in \mathbb{R}$, let $K(c) :=$ subcomplex with simplices σ with $f(\sigma) \le c$, and their faces
3. Proposition : Suppose $\exists !$ simplex α s.t. $f(\alpha) \in (a, b]$.
   (i) If α is non-critical, then either $K(b) = K(a)$ or $K(b)$ simplicially collapses to $K(a)$.
   Thus $K(b) \simeq K(a)$
   (ii) If α is critical, then $K(b) \simeq K(a)$ with a d-cell attached, where $d = \dim \alpha$.

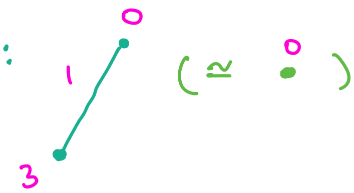
Upshot : We'll build K one step at a time using the K(c)'s, and the htpy type only changes when we add a critical simplex.

Eg :

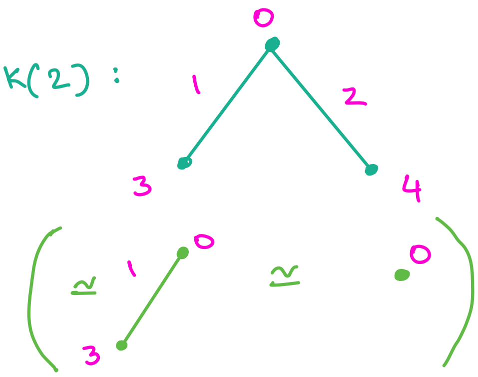


K(0) :

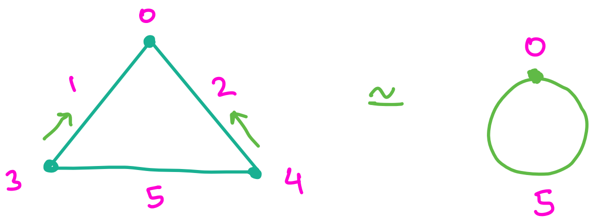
K(1) :



K(2) :



K(5) :



III Discrete Vector Fields

In practice, it is often hard to explicitly construct Discrete Morse Functions.

However, we saw that what really matters is the (pairs of) non-critical simplices.

This idea is captured by the concept of Discrete Vector Fields

Defn: (Discrete Vector Field)

Set of pairs (α, β) of simplices of K s.t. $\alpha \subset \beta$ codim 1

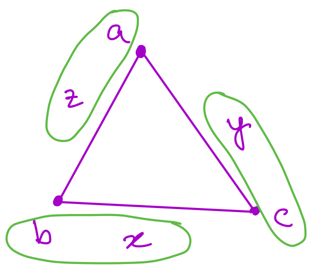
and every simplex σ is in atmost one such pair

Defn: (Gradient Vector Field)

If f is a Discrete Morse function, then the pairs of non-critical simplices form a discrete vector field. We call this the Gradient Vector Field of f.

Q: When does a discrete vector field arise from a discrete morse function?

Non-example



$\{(a,z),(b,x),(c,y)\}$ is NOT a gradient vector field

Problem: If it were, we'd have

$$f(a) > f(z) > f(b) > f(x) > f(c) > f(y) > f(a)$$

non-crit pair

discrete morse condition

Essentially, we can't have "cycles" amongst the pairs

$$a \subset z \supset b \subset x \supset c \subset y \supset a$$

Thm: If V is a cycle-free Discrete Vector Field, then it is the gradient vector field of some Discrete Morse Function.

Example: k-skeleton of an n-simplex

The simplices of an n-simplex Δ^n correspond to subsets of $[n]_0 = \{0, 1, 2, \ldots, n\}$.

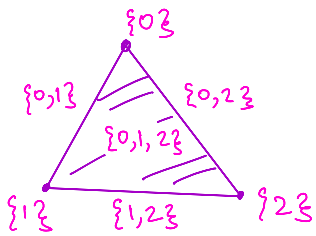
The k-skeleton, i.e. the dim $\leq k$ part, corresponds to subsets of size $\leq k+1$

Thm: $(\Delta^n)^k \simeq \bigvee_{\binom{n}{k+1}} S^k$

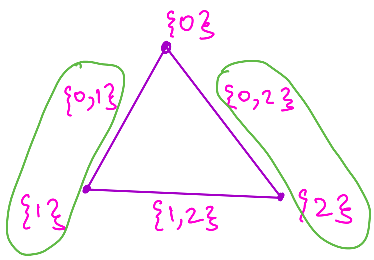
Pf: For every subset $S \subset [n]_0$ s.t. $0 \notin S$ and $|S| \le k$, pairing S with $S \cup \{0\}$ creates a cycle-free discrete vector field.

The critical simplices are $\{0\}$ and all $(k+1)$-subsets of $\{1, 2, \ldots, n\}$. Thus the result follows.

Eg: 2-simplex

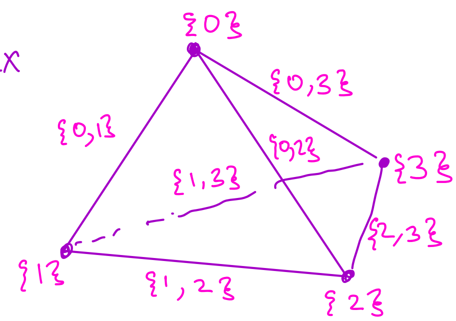


1-skeleton



Green pairs form a gradient vector field.

Eg: The 1-skeleton of a 3-simplex



A gradient vector field: